The Application of a New Class of Equal-Ripple Functions to Some Familiar Transmission-Line Problems

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Summary—Chevyshev's procedure for determining equal-ripple rational functions with preassigned poles is extended to functions with double-valued singularities. As long as the number of elements is small, design equations for the class of transmission-line filters consisting of shunt-resonant elements spaced a quarter wavelength apart are readily obtained by identifying the unknown coefficients with those of the desired equal-ripple function. This is carried out in some detail for three and four element filters and applied to the design of broad-band stub supports and quarter-wave-spaced broad-band TR tubes. Experimental confirmation is presented.

Introduction

The general synthesis of transmission-line filters consisting of short-circuited, quarter-wave stubs spaced a quarter-wavelength apart on sections of transmission line, each of undetermined characteristic impedance, has been considered by Jones.[1] He has shown that the insertion loss function P_L of a symmetrical filter of this type will take the form $P_L=1+Q_{n+1}^2(\omega)/(1+\omega^2)^n$, where Q_{n+1} is an even or odd polynomial of degree $n+1$ in ω with real coefficients and n is the number of quarter-wavelength sections of transmission line. He also pointed out how equal-ripple performance can be achieved for arbitrary bandwidth and tolerance by means of an ingenious potential analogy used by Grinich[2] and Bennett[3] for this purpose. This transformation is rather involved with the result that Jones limited has calculations of the coefficients of $Q_{n+1}(\omega)$ to a single bandwidth.

The general case treated by Jones is of considerable interest because it includes the problem of the design of optimum broad-band stubs,[4,5] broad-band TR tubes[6] and some specialized filters.[7–9]

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[1] E. M. T. Jones, "Synthesis of wide-band microwave filters to have prescribed insertion loss," 1956 IRE Convention Record, pt. 5, pp. 119–128.

[2] V. H. Grinich, "On the approximation of arbitrary phase-frequency characteristics," Electronics Research Lab., Stanford University, Stanford, Calif., Tech. Rept. No. 61, pp. 82–87; May 1, 1963.

[3] B. J. Bennett, "Synthesis of electric filters with arbitrary phase characteristics," 1953 IRE Convention Record, pt. 5, pp. 19–26.

[4] G. L. Ragan and R. M. Walker, "Rigid Transmission Lines," M.I.T. Rad. Lab. Ser., McGraw-Hill Book Company, Inc., New York, N. Y., vol. 9, pp. 173–176; 1948.

[5] C. E. Muehe, "Quarter-wave compensation of resonant discontinuities," IRE Trans. on Microwave Theory and Techniques (*Correspondence*), vol. MTT-7, pp. 296–297; April, 1959.

[6] W. C. Caldwell, "Bandpass T.R. Tubes," M.I.T. Rad. Lab. Ser., McGraw-Hill Book Company, Inc., New York, N. Y., vol. 14, pp. 67–96; 1948.

[7] A. W. Lawson and R. M. Fano, "The Design of Microwave Filters," McGraw-Hill Book Company, Inc., New York, N. Y., vol. 9, pp. 688–690; 1948.

The Problem and Solution

Now $\omega=-\cot\theta$, where θ is the common electrical length of the stubs and line lengths in the filter under consideration, *i.e.*, $\theta=2\pi l/\lambda_g$. If one uses $x=-\cos\theta$ for a frequency variable instead of ω, it is readily shown by means of the substitution, $w^2=\cot^2\theta=\cos^2\theta/(1-\cos^2\theta)=x^2(1-x^2)$ that $P_L=1+P_{n+1}^2(x)/(1-x^2)$, where $P_{n+1}(x)$ is an even or odd polynomial in x of degree $n+1$ with real coefficients.

Thus the problem of designing for equal-ripple performance reduces to finding even and odd polynomials $P_n(x)$ so that $P_n(x)/\sqrt{1-x^2}$ oscillates between ± 1 exactly $n+1$ times in a prescribed interval $-1<-x_c\lessapprox x\lessapprox x_c<1$.

In general, this problem may be solved either by use of the transformation employed by Grinich[2] and Bennett[3] or by adding that constant to the appropriate rational Chebyshev function[10] which converts it into a perfect square. Both of these procedures determine $P_n(x)$ by first determining $P_n^2(x)$. As is shown in the Appendix, the "primitive" equal-ripple rational functions employed by Chebyshev in the solution of his problem may be expressed in terms of more primitive, irrational functions. These are used to solve the problem directly. Finally $P_n(x)$[11] is given then by

$$2P_n(x) = (1+\sqrt{1-x_c^2})T_n(x/x_c) - (1-\sqrt{1-x_c^2})T_{n-2}(x/x_c), \qquad (1)$$

where $T_n(x)$ is the familiar Chebyshev polynomial of nth degree.

Although a general proof is given in the Appendix, the reader may wish to verify that $P_n(x)/\sqrt{1-x^2}$ equals $+1$ when $x=x_c$ equals ± 1 for $x=-x_c$ and equals 0 or ± 1 for $x=0$, depending on the precise value of n.

This solution of the approximation problem in closed form permits the exact determination of the optimum performance available from this class of transmission-

[8] S. B. Cohn, "Parallel-coupled transmission-line resonator filters," IRE Trans. on Microwave Theory and Techniques, vol. MTT-6, pp. 223–237; April, 1958.

[9] G. L. Matthaei, "Interdigital band-pass filters," IRE Trans. on Microwave Theory and Techniques, vol. MTT-10, pp. 479–491; November, 1962.

[10] P. L. Chebyshev, "Oeuvres," St. Petersbourg, Tome 1, pp. 313–331; 1899.

[11] These polynomials are readily shown to be unique and optimum by the familiar argument that an nth degree polymonial with $n+1$ roots must vanish identically.

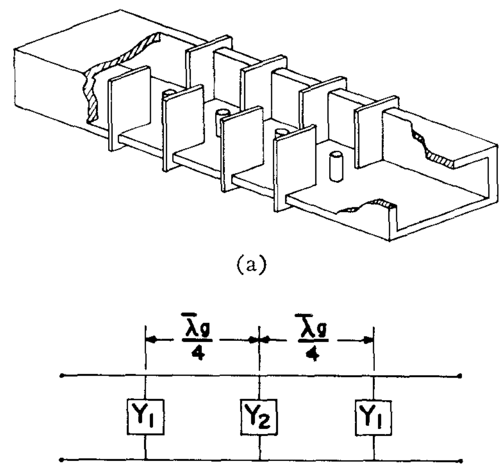
line filter and so permits the filter designer to select without approximation the number of filter elements required to achieve a specified selectivity for a given passband width and tolerance. It also allows the direct determination of the design parameters of a number of useful microwave devices by the method of undetermined coefficients if the number of unknowns is small. Of course, in the general case, the synthesis procedure of Jones[1] is available.

Using the frequency variable x instead of ω not only simplifies the problem but also simplifies the solution. In terms of ω, (1) would involve all the Chebyshev polynomials of the form $T_{n-2k}(\omega)$. That the denominator of P_L, expressed in terms of x, is independent of the number of filter sections is associated with the fact that only one stub is required to make the problem determinant. Additional stubs result in more unknown impedances than the number of independent coefficients in P_L. Moreover, if $P_{n+1}(x)$ is divisible by x^2-1, then no shunt elements are required in the realization.[12] This situation illustrates the fact that the approximation problem is most readily discussed in terms of x.

On the other hand, the frequency variable ω is very useful for synthesis realizability considerations. Because of the multivalued relationship between ω and x, x cannot be used conveniently as the frequency variable in a synthesis procedure based on positive realness in the sense of Brune.[13] For this reason the writer prefers to carry the analysis of a given problem through in terms of both $\cos\theta$ and $\sin\theta$ since this permits ready transformation from one variable to the other.

Special Cases

The first application to be considered is that of the structure of Fig. 1(a). Here we have shunt elements consisting of inductive irises which have been tuned to the midband frequency by means of the shunt capacities. When these resonant elements are spaced a quarter wavelength apart, the present theory is certainly applicable over frequency bands of the order of 20 per cent or less. This follows from the fact that the shunt susceptance of a resonant iris will certainly closely approximate, as a function of frequency, the shunt susceptance of a quarter-wave stub of suitable characteristic admittance. Thus having calculated the stub admittance, one may then determine the corresponding iris Q. Of course, the iris filter will not have the theoretical response near the frequency where the irises are a half wavelength apart, but, for many applications, this is of secondary importance.



$$Y_1=-ja\cot\theta\;;\quad Y_2=-j2b\cot\theta$$

(b)

Fig. 1—Filter schematics.

To make the problem determinant the additional requirement will be imposed that all of the quarter-wavelength sections have the same characteristic impedance as the generator and load (this is often a practical condition). This restriction reduces the original problem of Jones to one which is precisely determinant since the number of unknowns now just equals the number of defining equations. However, it is now no longer possible to prove general physical realizability. Experience with the solution of a number of cases indicates, nevertheless, that the ideal response is realizable with the required structure in all but the very broad-band extremes.

Let us now consider the determination of the shunt susceptances of the three resonator problem of Fig. 1(b). We may think of this as three stubs of characteristic admittance, a, $2b$, and a, each spaced one quarter wavelength apart on a uniform transmission line of unity characteristic impedance. Our first problem is the determination of $P_L=P_{\text{avail}}/P_{\text{load}}$ in terms of a and b.

If v_o, i_o, v_i and i_i are the output voltage and current of a network and input voltage and current of the network, respectively, then the $ABCD$ or transfer matrix of the network is the matrix of the coefficients of the equations,

$$v_i = Av_o + jBi_o$$
$$i_i = jCv_o + Di_o,$$

giving the inputs in terms of the outputs. Now it is well known that if the transfer matrix of one half of a symmetrical network is written in the form,

$$\begin{pmatrix} A & jB \\ jC & D \end{pmatrix},$$

[12] H. J. Riblet, "General synthesis of quarter-wave impedance transformers," IRE Trans. on Microwave Theory and Techniques, vol. MTT-5, pp. 36–43; January, 1957.

[13] O. Brune, "Synthesis of a finite two-terminal network whose driving point impedance is a prescribed function of frequency," *J. Math. and Phys.*, vol. 10, pp. 191–236; October, 1931.

the insertion loss P_L of the complete network, terminated at both ends in unity impedances, is given by $1+(BD-AC)^2$. Thus we are concerned with $\mathcal{E}=BD-AC$.

To determine the transfer matrix of the bisected network, we require the transfer matrices of the three elements of which it is composed. For a length of transmission line of unity characteristic impedance,

$$v_i = \cos\theta v_o + j\sin\theta i_o$$
$$i_i = j\sin\theta v_o + \cos\theta i_o,$$

while for the shunt stub of characteristic admittance a,

$$v_i = v_o$$
$$i_i = -j\frac{a\cos\theta}{\sin\theta}v_o + i_o,$$

where $\theta=\pi\bar{\lambda}_g/2\lambda_g$ and λ_g and $\bar{\lambda}_g$ are the variable and midband guide wavelengths, respectively.

Thus, the transfer matrix for the half of the network on the right in Fig. 1(b) is obtained from the matrix product,

$$\begin{pmatrix} 1 & 0 \\ \dfrac{-jbc}{s} & 1 \end{pmatrix}\begin{pmatrix} c & js \\ js & c \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \dfrac{-jac}{s} & 1 \end{pmatrix},$$

where for brevity $c=\cos\theta$ and $s=\sin\theta$. Notice that the square of the left-hand matrix above is just

$$\begin{pmatrix} 1 & 0 \\ \dfrac{-j2bc}{s} & 1 \end{pmatrix},$$

so that we have a true bisection of the network. When this matrix product is evaluated, we obtain

$$\begin{pmatrix} c(1+a) & js \\ j\left[s - \dfrac{(b+a+ab)c^2}{s}\right] & c(1+b) \end{pmatrix}.$$

Thus,

$$\mathcal{E}(s,c) = sc(b-a) + \frac{(b+a+ab)(a+1)c^3}{s}. \quad (2)$$

Now since P_L is given most readily in terms of a function of c divided by $\sqrt{1-c^2}$, we write $\mathcal{E}(s,c)=\mathcal{E}(c)/\sqrt{1-c^2}=\mathcal{E}(c)/s$. Then $\mathcal{E}(c)=(b+1)(a^2+2a)c^3-(a-b)c$. Thus all possible responses of the network are expressible in the form $P_L=1+\mathcal{E}^2(c)/s^2$.

The equal-ripple response which permits an exact identification of coefficients has the form,

$$P_L = 1 + h^2\left\{\frac{(1+\sqrt{1-x_c^2})T_3(x/x_c) - (1-\sqrt{1-x_c^2})T_1(x/x_c)}{2\sqrt{1-x^2}}\right\}^2, \quad (3)$$

where h gives the tolerance on insertion loss and x_c is the maximum value of x where this tolerance is achieved. If we then put $x=c$, $x_c=\mu$, $\sqrt{1-x^2}=s$, $T_3(x)=4x^3-3x$ and $T_1(x)=x$, we require that

$$\frac{(b+1)(a^2+2a)c^3-(a-b)c}{s}$$
$$= h\frac{(1+\sqrt{1-\mu^2})(4\dfrac{c^3}{\mu^3}-3\dfrac{c}{\mu})(-(1-\sqrt{1-\mu^2})\dfrac{c}{\mu}}{2s}.$$

Now equating the coefficients of c^3 and c, we find

$$a(a+2)(b+1) = 2h(1+\sqrt{1-\mu^2})/\mu^3$$
$$a-b = h(2+\sqrt{1-\mu^2})/\mu.$$

A three-element filter was designed on this basis and its input VSWR is compared with the theory in Fig. 2.

In the application of this theory to cascades of resonant elements in which n is even, such as the familiar four element TR tube, it should be observed that the function which varies $n+1$ times between ± 1 over the range $-1\leq x\leq 1$,

$$\frac{(a+\sqrt{a^2-1})T_n(x) - (a-\sqrt{a^2-1})T_{n-2}(x)}{2\sqrt{a^2-x^2}}, \quad (4)$$

as derived in the Appendix, may be used to construct the optimum, even, equal-ripple functions having a double zero at zero. For this it is observed that the numerator of (4) is even if n is even. Then if x_0 is the zero of this function nearest to the origin, replacing x^2 above by $x'^2+x_0^2$, yields the desired optimum response function since this transformation maps the values of x between x_0 and 1 into the values of x' between 0 and $\sqrt{1-x_0^2}$ without altering the total number of zeros in the pass band of x'.

Although no general procedure is known for the determination of x_0, an explicit formula can be derived when $n=4$. If $Z=a+\sqrt{a^2-1}$, we have to solve

$$ZT_4(x) - Z^{-1}T_2(x) = 0. \quad (5)$$

But $T_4(x)=2T_2^2(x)-1$ so that (5) becomes $2ZT_2^2(x)-Z^{-1}T_2(x)-Z=0$ or

$$T_2(x_0) = \cos 2\cos^{-1}x_0 = \frac{1-\sqrt{1+8Z^4}}{4Z^2}. \quad (6)$$

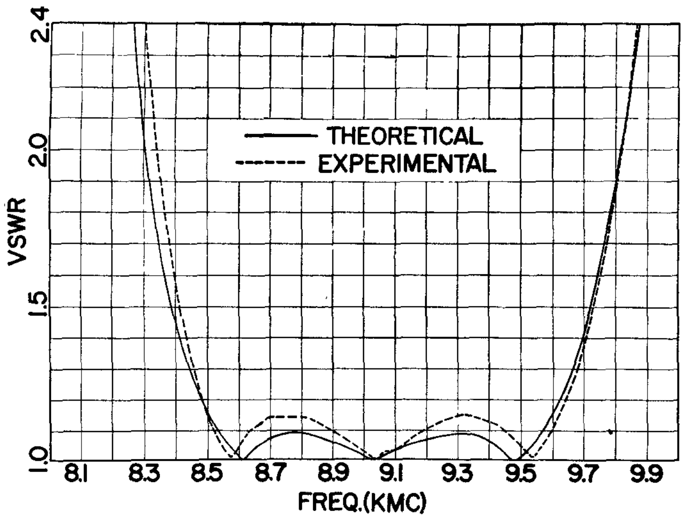


Fig. 2—Response of three-element filter.

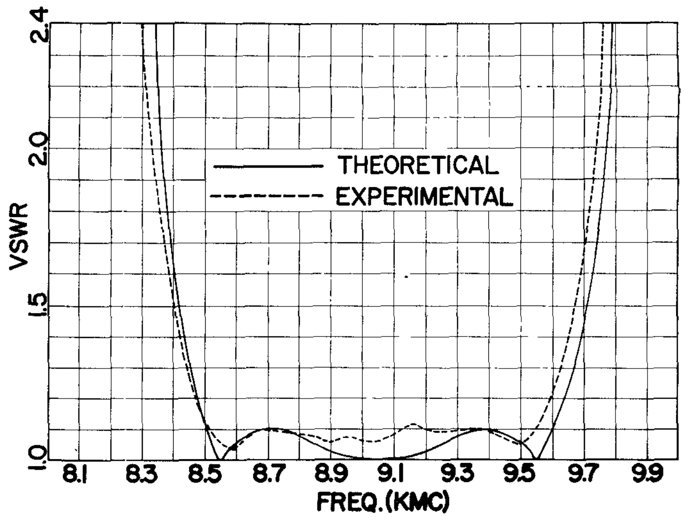


Fig. 3—Response of four-element filter.

The displacement of the zero also changes the range of x over which equal-ripple performance is observed. In discussing this aspect of the problem, it is convenient to introduce

$$R_n(y^2) = ZT_n(y) - Z^{-1}T_{n-2}(y).$$

Then $R_n(x^2+x_0^2)/2\sqrt{a^2-x_0^2-x^2}$ has a double zero at $x=0$ and varies between ± 1 over the range $\pm\sqrt{1-x_0^2}$. If we put $x=\sqrt{a^2-x_0^2}x'$, then

$$R_n[(a^2 - x_0^2)x'^2 + x_0^2]/2\sqrt{a^2 - x_0^2}\sqrt{1 - x'^2} \qquad (7)$$

has the desired form in the denominator and varies between ± 1 over the range

$$\pm\sqrt{1 - x_0^2}/\sqrt{a^2 - x_0^2}.$$

The limiting values $\pm\mu$ of $x' = \cos\theta$ are then given by $\mu = \sqrt{1-x_0^2}/\sqrt{a^2-x_0^2}$. This is determined by the given filter bandwidth. Our problem is the determination of a and x_0, but for given n, x_0 is a function of a. Hence μ is a function of a. Given μ, a is first determined from this functional relationship. Then x_0 is found from the function relating it to a. The required equal-ripple function with a double zero at 0 is now provided by (7).

This procedure was used to determine the iris Q's for a four element TR tube to operate with a VSWR$\leq$1.1 over the band from 8500 to 9600 Mc in WR90 waveguide. Fig. 3 compares the measured input standing wave ratio with that which was calculated.

If the transmission line terminates in impedance sections rather than shunt resonant elements, the theory is applicable to the design of broad-band stub supports for coaxial line. The usual case is pictured in Fig. 4(a). It has been discussed by Pound[14] and Muehe.[5] The transfer matrix of the bisected cascade is given by

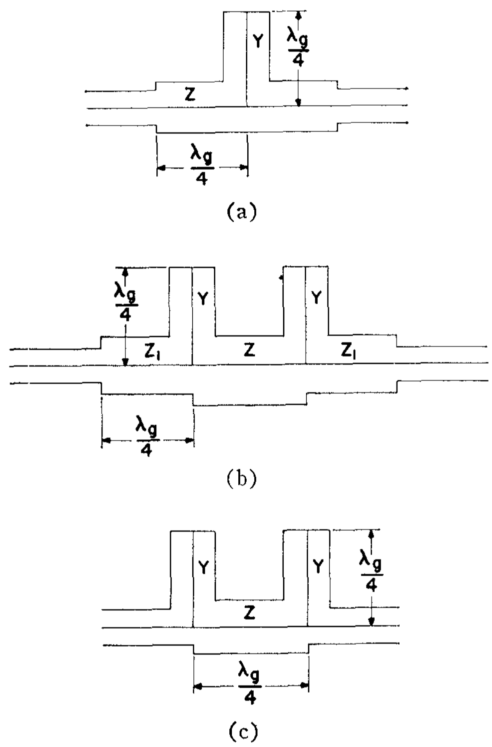


Fig. 4—Stub schematics.

$$\begin{pmatrix} 1 & 0 \\ -j\dfrac{Yc}{2s} & 1 \end{pmatrix}\begin{pmatrix} c & jsZ \\ \dfrac{js}{Z} & c \end{pmatrix}.$$

For equal-ripple performance over the range $\pm\mu$ with a maximum variation in P_L of h^2, we find that

$$\frac{Y}{2} - \frac{Y}{2}Z^2 + \frac{1}{Z} - Z = 2h(1 + \sqrt{1 - \mu^2})/\mu^3$$

$$-\frac{Y}{2}Z^2 + \frac{1}{Z} - Z = h(2 + \sqrt{1 - \mu^2})/\mu. \qquad (8)$$

[14] R. V. Pound, "Stub Supports in 7/8 in. Coaxial Line," M.I.T. Radiation Lab., Cambridge, Rept. No. 232; May 19, 1942.

For $Y=1$, this is equivalent to the condition given by Pound[14] and, in general, is equivalent to (4) and (5) of Muehe.[5]

The calculations for the symmetrical two stub case of Fig. 4(b) are somewhat more complicated. The equations giving Y, Z and Z_1 in terms of μ and h are

$$\frac{Z}{2Z_1^2} - \frac{Z_1^2}{2Z} = h$$

$$Y + \frac{Y^2 Z}{2} = \frac{h}{\mu^4}\{\mu^4 - 5\mu^2 + 4 + (4 - 3\mu^2)\sqrt{1-\mu^2}\}$$

$$Z_1 - \frac{1}{Z_1} + \frac{1}{2}\left(Z - \frac{1}{Z}\right) + Z_1 Y(Z + Z_1)$$

$$+ \frac{Y^2 Z_1^2 Z}{2} - \frac{YZ}{Z_1} = \frac{h}{\mu^2}(2\mu^2 - 5 - 3\sqrt{1-\mu^2}). \quad (9)$$

From a practical point of view, the stub impedance Y cannot vary too greatly from unity. On the other hand, the behavior of P_L in the vicinity of its singularity at $c=1$ is proportional in a general way to the total amount of the shunt susceptance in the cascade. A consideration of the form P_L will show that this susceptance must increase rapidly with increasing n. How this affects the broad-band performance of multiple stub support is indicated in Fig. 5, where for two fixed values of Y the available VSWR tolerance is plotted as a function of the bandwidth of the stub support. A two-stub support makes possible a lower VSWR for certain bandwidths but for larger bandwidths a single stub is superior. For example, over a one-octave band, corresponding to $\cos\theta_c = 0.5$, a two-stub arrangement in which the stubs have the same characteristic impedance as the terminating impedance has a maximum VSWR of 1.04 while a single stub of the same impedance gives a maximum VSWR of 1.1. On the other hand, over a two octave band, corresponding to $\cos\theta_c = 0.809$, a single stub of relative admittance 0.5 is superior to a double stub with the same relative admittance.

The shortest compensated stub arrangement is that shown in Fig. 4(c). Of course, it has the same response as the single stub of Fig. 4(a). For it,

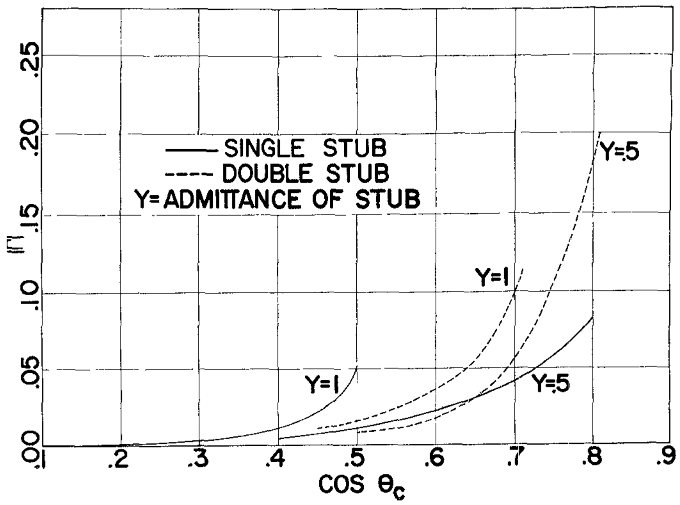


Fig. 5—Comparison of stub responses.

$$Z - \frac{1}{Z} = -2h$$

$$ZY^2 + 2Y = 2h\left(\frac{1+\sqrt{1-\mu^2}}{\mu^2} - 1\right). \quad (10)$$

For some applications, μ and Y are given while Z and h are required. If one writes $\alpha = (1+\sqrt{1-\mu^2})/\mu - 1$, then

$$Z = (\sqrt{Y^2(1+\alpha)+\alpha^2} - Y)/(Y^2+\alpha).$$

Appendix

Chebyshev[12] solved the problem of determining equal-ripple rational functions having fixed singularities by a method which is analytically equivalent to that proposed later by Bernstein.[15] Bernstein defined an angle δ by means of the equations

$$\cos\delta = \frac{ax-1}{a-x}; \qquad \sin\delta = \frac{\sqrt{a^2-1}\sqrt{1-x^2}}{a-x},$$

where $x=\cos\phi$. He argued then that as ϕ goes from zero to π so does δ and constructed the required equal-ripple functions by evaluating expressions of the form $\cos(n\phi+\delta_1+\delta_2+\cdots)$ where the δ's are defined by various a's.

Double-valued singularities may be included in this procedure since, if

$$e^{i\delta_+} = \cos\delta_+ + i\sin\delta_+ = \frac{(a+\sqrt{a^2-1})e^{i\phi} + (a-\sqrt{a^2-1})e^{-i\phi} - 2}{2(a-x)}, \quad (11)$$

$$e^{i\delta_+/2} = \frac{(\sqrt{a+1}+\sqrt{a-1})e^{i\phi/2} - (\sqrt{a+1}-\sqrt{a-1})e^{-i\phi/2}}{2\sqrt{a-x}}. \quad (12)$$

[15] S. Bernstein, "Leçons sur les properiétés extremals et la meileure approximation des fonctions analytiques d'une variable reélle," Gauthier-Villars et Cie, Paris, France, pp. 1–12; 1926.

Now to assure that all radicals have real positive values, in the interval $-1 \leq x \leq 1$, it must be assumed that $a > 1$. Then for singularities on the negative real axis, we replace a by $-a$ and define

$$e^{i\delta_-/2} = \frac{(\sqrt{a+1} + \sqrt{a-1})e^{i\phi/2} + (\sqrt{a+1} - \sqrt{a-1})e^{-i\phi/2}}{2\sqrt{a+x}}. \tag{13}$$

Thus, in the factorization of the denominator of the desired equal-ripple function, it is assumed that each factor is positive in the range $-1 \leq x \leq 1$. Now (12) is a "primitive" function in terms of which the problem of Chebyshev and our problem are readily solved.

We show that δ_+ and δ_- increase monotonically from 0 to π (except for $2K\pi$) as ϕ increases from 0 to π. In the first place, both δ_+ and δ_- are real for real ϕ, since

$$\left| e^{i\delta_\pm/2} \right|^2 = \frac{(a + \sqrt{a^2-1}) + (a - \sqrt{a^2-1}) \mp 2(e^{i\phi} + e^{-i\phi})}{2(a \mp x)} = 1. \tag{14}$$

[For future reference, notice that the second equality of (14) does not require a to be real.] From (12) and (13), when $\phi = 0$, $e^{i\delta_\pm/2} = 1$, while $e^{i\delta_\pm/2} = i$ when $\phi = \pi$. Thus, within integral multiples of 2π, $\delta_\pm$ varies from 0 to π when ϕ does. Now $\sin(\delta_\pm/2) = \sqrt{a \mp 1}\sin(\phi/2)/\sqrt{a \mp \cos\phi}$, so that $\sin(\delta_\pm/2)$ is positive whenever $\sin(\phi/2)$ is, since $a > 1$. Thus $\delta_\pm$ increases monotonically from $2K\pi$ to $(2K+1)\pi$ whenever ϕ goes from 0 to π.

Now consider

$$\cos\left(n\phi + \frac{\delta_+ + \delta_-}{2}\right) = \text{Re}\left\{e^{i(n\phi + (\delta_+ + \delta_-)/2)}\right\}. \tag{15}$$

Clearly, except for multiples of 2π,

$$n\phi + \frac{\delta_+ + \delta_-}{2}$$

increases from 0 to $(n+1)\pi$ as ϕ goes from 0 to π and so (15) has the desired equal-ripple performance.

From (12) and (13),

$$e^{i((\delta_+ + \delta_-)/2)} = \frac{(a + \sqrt{a^2-1})e^{i\phi} - (a - \sqrt{a^2-1})e^{-i\phi}}{2\sqrt{a^2 - x^2}}.$$

Thus

$$\text{Re}\left\{\frac{(a + \sqrt{a^2-1})e^{i(n+2)\phi} - (a - \sqrt{a^2-1})e^{in\phi}}{2\sqrt{a^2 - x^2}}\right\}$$

is the desired equal-ripple function. But $\text{Re}(e^{in\phi}) = T_n(x)$ so that

$$\frac{(a + \sqrt{a^2-1})T_{n+2}(x) - (a - \sqrt{a^2-1})T_n(x)}{2\sqrt{a^2 - x^2}}$$

is the required equal-ripple function of degree $n+2$.

The arguments given above are no longer valid if a is complex. There is some interest, however, in the case of singularities which occur in complex pairs.[3,16,17] In the interest of completeness this situation will now be discussed. Define δ by

$$e^{i\delta} = \frac{(\sqrt{a+1} + \sqrt{a-1})\, e^{i\phi/2} - (\sqrt{a+1} - \sqrt{a-1})e^{-i\phi/2}}{2\sqrt{a-x}} \tag{16}$$

with $a = \alpha + i\beta$ and $\beta \neq 0$, and define $\bar{\delta}$ by

$$e^{i\bar{\delta}} = \frac{(\sqrt{a^*+1} + \sqrt{a^*-1})e^{i\phi/2} - (\sqrt{a^*+1} - \sqrt{a^*-1})e^{-i\phi/2}}{2\sqrt{a^* - x}}, \tag{17}$$

where a^* is the complex conjugate of a. Now $\delta + \bar{\delta}$ varies from 0 to π (except for multiples of 2π) as ϕ does, so that $e^{i(\delta+\bar{\delta})}$ can be used to construct equal-ripple functions after the manner already indicated.

[16] C. B. Sharpe, "A general Chebyshev rational function," PROC. IRE, vol. 42, pp. 454–457; February, 1953.

[17] D. Helman, "Synthesis of electric filters with arbitrary phase characteristics," IRE TRANS. ON CIRCUIT THEORY (*Correspondence*), vol. CT-2, pp. 217–218; June, 1955.

To demonstrate this, first notice that $|e^{i(\delta+\bar{\delta})}|=1$. This follows from the fact that

$$e^{i(\delta+\bar{\delta})}\cdot[e^{i(\delta+\bar{\delta})}]^* = [e^{i\delta}\{e^{i\bar{\delta}}\}^*]\cdot[e^{i\bar{\delta}}\{e^{i\delta}\}^*],$$

and each of these factors has the form of the middle term of (13), where a is not required to be real for the second equality to hold.

Now $e^{i(\delta+\bar{\delta})}$ can be expressed with real coefficients as

$$e^{i(\delta+\bar{\delta})} = \frac{(r+\sqrt{r^2-1})e^{i\phi} - 2\alpha/r + (r-\sqrt{r^2-1})e^{-i\phi}}{2\sqrt{x^2-2\alpha x+\alpha^2+\beta^2}}, \quad (18)$$

where r is defined as the semi-major axis of the ellipse passing through (α, β) with foci at ± 1.[18] The quantities under the radical signs are all positive so that (18) is well defined and valid for all α if $\beta \neq 0$. If $\beta = 0$ and $\alpha > 1$ then (17) reduces to (11) and if $\alpha < 1$, (17) reduces to the square of (13). Finally

$$\sin(\delta+\bar{\delta}) = \frac{\sqrt{r^2-1}\sin\phi}{2\sqrt{x^2-2\alpha x+\alpha^2+\beta^2}}$$

so that $\delta+\bar{\delta}$ goes from 0 to π as ϕ does since $\sin(\delta+\bar{\delta})$ is positive everywhere in between, regardless of the sign of α.

[18] For the transformations involved see Bernstein.[15]

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A Broad Tunable Bandwidth Traveling-Wave Maser

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***Summary*—A new type of traveling-wave maser (TWM) has been developed, employing the meander line as the slow-wave circuit and rutile as the active maser crystal. This amplifier has achieved net gains in excess of 23 db across the band from 2.0 to 3.0 Gc, with an over-all noise temperature of 8° ± 2°K. This marks the first time that rutile with a dielectric constant of 220 has been coupled to a slow-wave circuit. The maser material exhibited inversion ratios of 10:1 and saturated at an input signal of −47 dbm. In addition to the maser work, a ferrite material investigation was conducted, which led to the development of a gadolinium substituted yttrium iron garnet (YIG) as the ferrite isolator. Various concentrations of the gadolinium in YIG were investigated as ferrite isolators at 4.2°K and were found to have lower forward losses than pure YIG at *S* band.**

Introduction

THE THEORY and advantages of the traveling-wave maser over the cavity maser has been well established and given in the literature [1], [2]. Various slow-wave structures, such as the comb structure [1], Karp structure [3], meander line [4]–[6] and the dielectrically loaded waveguides [7], have been used to obtain the low group velocities necessary for traveling-wave maser action. This paper specifically describes the design of an *S*-band rutile loaded meander line. The energy levels and crystal axis orientations of rutile are discussed since this is the first time this material has been used at such low frequencies.

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The initial desire for a large tunable bandwidth eliminated structures such as the comb and Karp because of their highly resonant, narrow bandwidth characteristics. A slow-wave structure which exhibited wide dispersion characteristics, relatively low impedance variations as a function of frequency and low insertion losses was needed. Investigations showed the meander line to possess these characteristics.

Rutile was selected as the active paramagnetic material because of its very high inversion ratio, high dielectric constant, high signal power saturation levels and the ability to obtain large single crystals with accurate crystal axis alignment.

Characteristics of Rutile [8]

Ruby has been generally used as the active material in TWM's. For this application, certain improved characteristics were desired, and rutile was considered as superior to ruby for TWM design. The pertinent characteristics of these two materials are compared in Table I. Note the advantages of rutile's lower Q_m and higher dielectric constant. Also, the zero field splitting for rutile is large, which contributes to a higher inversion ratio. In addition rutile saturates at a relatively high input level. Rutile was selected for these advantages,